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ON STRUCTURAL COMPLETENESS OF THE INFINITE-VALUED ŁUKASIEWICZ'S PROPOSITIONAL CALCULUS

This is an abstract of my paper "On structural Completeness of many-valued logics" submitted to **Studia Logica**.

I. In the paper the quasi-structural consequence generated by the infinite-valued Łukasiewicz's calculus $\langle R_{0*}, A_\infty\rangle$ is examined. The notions of consequence operations $Sb(X), Cn(R_0, Sb(X))$ and $Cn(R_{0*}, X)$, where $R_0 = \{r_0\}$ and $R_{0*} = \{r_0, r\}$ (r_0 is the modus ponens rule and r_* is the substitution rule), are defined in [2]. Recall that a consequence Cn is quasi-structural ($Cn \in Sb - Struct$) iff there exists a consequence $Cn_1 \in Struct$ such that $Cn = Cn_1 Sb$. The structural consequence generated by a matrix M is denoted by $\overrightarrow{M}$, and the symbol $E(M)$ stands for the set of all valid formulas in this matrix ($E(M) = \overrightarrow{M}(0)$). For every matrix M we have:

Theorem 1. *$\overrightarrow{M}(Sb(X)) = \bigcap\{E(N) : N \subseteq M$ and $X \subseteq E(N)\}$ for every $X \subseteq S$.*

Proof. Let $M = \langle |M|, |M|^*; f_1, \ldots, f_n\rangle$. Inclusion ($\subseteq$) is obvious. ($\supseteq$). If $\overrightarrow{M}(Sb(X)) = S$, then the inclusion is also true. Suppose that $\overrightarrow{M}(Sb(X)) \neq S$. Hence $V = \{v : At \to M;\ h^v(SbX) \subseteq |M|^*\} \neq 0$. For every $v \in V$ let a submatrix M_v of M be defined as follows: $M_v = \langle h^v(S), h^v(S) \cap |M|^*;\ f_1, \ldots, f_n\rangle$. We shall show that $X \subseteq E(M_v)$ for every $v \in V$. Let $w : At \to |M_v|$. There exists $e : At \to S$ such that $w = h^v e$. Thus $h^w(SbX) = h^v(h^e(SbX)) \subseteq h^v(SbX) \subseteq |M|^* \cap h^v(S) = |M_v|^*$. Hence $X \subseteq E(M_v)$.

Assume that $\alpha \notin \overrightarrow{M}(Sb(X))$. There exists $v \in V$ such that $h^v\alpha \notin |M|^*$.

Hence $\alpha \notin E(M_v)$. Thus inclusion $(\supseteq)$ is also true.

From Theorem 1 Wójcicki's theorem on degree of completeness of a strongly finite consequence (cf. [5]) follows immediately.

COROLLARY. *If Cn is strongly finite, then the degree of completeness of $Cn\ Sb$ is finite.*

PROOF. $Cn = \overrightarrow{N}_1 \cap \ldots \cap \overrightarrow{N}_k$ where N_i is a finite matrix for every $i \leqslant k$. Every finite matrix has a finite set of submatrices. By Theorem 1 the degree of completeness of $\overrightarrow{N}_i$ is finite.

II. Let $M_\infty = \langle Q \cap [0,1], \{1\}; c, a, k, e, n\rangle$ and $M_c = \langle [0,1], \{1\} : c, a, k, e, n\rangle$ be Łukasiewicz's matrices (Q is the set of all rational numbers). The symbol M_n denotes the n-valued Łukasiewicz's matrix. In [4] it is proved that:

$$Cn_{R_0, Sb(A_\infty)} \nleq \overrightarrow{M}_c \nleq \overrightarrow{M}_\infty \nleq \bigcap_{n \geqslant 2} \overrightarrow{M}_n.$$

For quasi-structural consequence generated by these matrices we have:

THEOREM 2. $Cn_{R_o^*}, A_\infty \nleq \overrightarrow{M}_c Sb = \overrightarrow{M}_\infty Sb = \bigcap_{n \geqslant 2} M_n Sb = \overrightarrow{\times_{n \geqslant 2} M_n} Sb.$

Observe that all these consequences are finite. Hence there exists a finite set $X \subseteq S$ such that $Cn(R_{0*}, A_\infty \cup X) \subsetneq \overrightarrow{M}_c(Sb(X))$. As known, M is a submatrix of M_n iff there exists $k \in N$ such that $k-1 | n-1$ and $M = M_k$. Thus from Theorem 1 we obtain the "topographic" theorem on Łukasiewicz's logics (cf. [3]): $\overrightarrow{M}_n(Sb(X)) = \bigcap\{E(M_k); X \subseteq E(M_k)$ and $M_k \subseteq M_n\}$ for every $X \subseteq S$. The following theorem follows directly from Theorem 1:

THEOREM 3. $\overrightarrow{M}_\infty(Sb(X)) = \bigcap\{E(M_n); X \subseteq E(M_n)\}$ *for every* $X \subseteq S$.

Moreover, observe that if $X \subseteq N$ and $\overline{\overline{I}} = \overline{\overline{N}}$, then $\bigcap_{i \in I} E(M_i) = E(M_\infty)$.

III. We recall the notion of structural Completeness (cf. [1]):

$Cn \in SCpl \Leftrightarrow Perm(Cn) \cap Struct \subseteq Der(Cn)$.
$Cn \in SCpl_F \Leftrightarrow Perm(Cn) \cap Struct \cap Fin \subseteq Der(Cn)$.

where Fin denotes the set of all rules with finite set of premisses. The symbol L_∞ stands for the Lindenbaum's matrix of $\langle R_{0*}, A_\infty\rangle$. The following lemma characterizes a structurally complete consequence in the set of all quasi-structural consequences.

LEMMA 1. $Cn \in Sb - Struct \Rightarrow \{Cn \in SCpl \Rightarrow \forall_{Cn_1 \in Sb-Struct}[Cn_1(0) = Cn(0) \Rightarrow Cn_1 \leqslant Cn]\}$.

LEMMA 2. *If* $M \subseteq L_\infty$, *then* $E(M) = E(M_2)$ *or* $E(M) = E(M_\infty)$.

Thus from THEOREM 1 we obtain the following theorem

THEOREM 4. $\overrightarrow{M}_\infty Sb \in SCpl(\overrightarrow{M}_\infty Sb \notin SCpl_F)$.

PROOF. We have

$$\overrightarrow{L}_\infty(Sb(X)) = \begin{cases} E(M_\infty) & \text{if } X \subseteq E(M_\infty) \\ E(M_2) & \text{if } X \subseteq E(M_2) \text{ and } X \not\subseteq E(M_\infty) \\ S & \text{if } X \not\subseteq E(M_2). \end{cases}$$

This consequence is structurally complete. From Theorem 3 we obtain that $\overrightarrow{M}_\infty Sb \notin SCpl$.

It follows directly from this theorem that $\langle R_{0*}, A_\infty\rangle \notin SCpl_F$. This result was first proved by dr T. Prucnal (unpublished).

IV. We examine now the positive infinite-valued Łukasiewicz's calculus $\langle R_{0*}, A^p_\infty\rangle$. Let M^p_n be the positive reduct of the n-valued Łukasiewicz's matrix.

THEOREM 5. $\bigcap_{n\geqslant 2} \overrightarrow{M}^p_n \in SCpl$.

The following corollary results from the above theorem and some results from [4]:

COROLLARY. $\langle R_0, Sb(A^p_\infty)\rangle \in SCpl_F - SCpl$, $\langle R_{0*}, A^p_\infty\rangle \in SCpl$.

As known, M is a submatrix of M^p_k iff there exists $m \leqslant k$ such that M is isomorphic to M^p_m. Thus by Theorem 1:

$$\bigcap_{n\geqslant 2} \overrightarrow{M}^p_n(Sb(X)) = \begin{cases} Cn(R_{0*}, A^p_\infty) & \text{if } X \subseteq Cn(R_{0*}, A) \\ E(M^p_m) & \text{if } m = max\{n \in N; X \subseteq E(M^p_n)\} \\ S^p & \text{if } X \not\subseteq E(M_2) \end{cases}$$

This consequence is structurally complete.

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